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THE BOUNDARY ELEMENT METHOD FOR CALCULATING VISCO-RIGID PLASTIC FLOWS†

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A method of solving the boundary-value problem for a visco-rigid plastic medium is considered which leads to the method of boundary elements.

1. STATEMENT OF THE PROBLEM

WE WILL use the following notation: R^3 is a space with fixed Euclidean coordinates x_1, x_2 and x_3; Ω is a region of R^3 of class C^1, i.e. the boundary $\partial\Omega$ is a two-dimensional manifold of class C^1 and the region Ω is situated locally on one side of $\partial\Omega$; $v = \{v_i\}$ is the velocity field in Ω, $e(v) = \{e_{ij}(v)\}$ is the strain rate tensor ($e_{ij} = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ $i, j = 1, 2, 3$); $\sigma = \{\sigma_{ij}\}$ is the stress tensor, and $s = \{s_{ij}\}$ is its deviator. For points from $\partial\Omega$, $\mathbf{n}$ denotes the unit external normal, $\mathbf{F} = \{F_i\}$ ($F_i = \sigma_{ij}n_j$) is the force density vector on $\partial\Omega$ (here and below summation over repeated indices is assumed); for any vector $\mathbf{a}$ applied at a point of $\partial\Omega$, a_n denotes its projection on the normal and $\mathbf{a}_t$ its tangential component, $|\mathbf{a}|$ is its length and $(\cdot, \cdot)$ the corresponding scalar product; for $e = \{e_{ij}\}$, $q = \{q_{ij}\}$ we assume $\langle e, q\rangle = e_{ij}q_{ij}$ and $|e| = (e_{ij}e_{ij})^{1/2}$; the measure in R^3 $dx = dx_1dx_2dx_3$, and dS is the measure on $\partial\Omega$ generated by dx; $T(\Omega)$ $[D(\Omega)]$ is the space of stress tensors $\sigma(x)$ [deviators $s(x)$] in Ω with components from $L^2(\Omega)$; $\boldsymbol{H}^1(\Omega)$ is the space of vector fields $\mathbf{v} = \{v_i\}$ in Ω such that v_i belongs to the Sobolev space $H^1(\Omega)$; $\boldsymbol{H}^{1/2}(\partial\Omega)$ is the space of vector fields $\mathbf{v} = \{v_i\}$ in $\partial\Omega$ such that v_i belongs to the Sobolev space $H^{1/2}(\partial\Omega)$; $H^{-1/2}(\partial\Omega)$ is the space of linear continuous functionals on $\boldsymbol{H}^{1/2}(\partial\Omega)$.

Consider a visco-rigid plastic medium, that is, a medium which is incompressible and for which the deviator of the stress tensor is defined [1] by the plastic potential

$$\varphi(e) = \tfrac{1}{2}\mu\,|e|^2 + \tau_s\,|e|$$

where μ is the coefficient of viscosity and τ_s is the yield point. We require the deviator s to belong to the subdifferential $\partial\varphi[e(v)]$, that is $s = \mu e(v) + \tau_s e(v)/|e(v)|$ if $e(v) \neq 0$, and $|s| \leq \tau_s$ if $e(v) = 0$.

For slow (quasi-stationary) processes, the real velocity and stress fields are defined by the following boundary-value problem.

Problem 1. In the region Ω, it is required to find the velocity field $\mathbf{v}$ and the stress tensor σ satisfying the following conditions:

1. the velocity field satisfies the condition $\operatorname{div}(\mathbf{v}) = 0$;
2. the equilibrium equations $\partial\sigma_{ij}/\partial x_j = 0$, $i = 1, 2, 3$ apply;
3. the equation of state of the medium $s \in \partial\varphi(e(v))$;
4. boundary conditions: the boundary $\partial\Omega$ consists of three parts with non-zero areas $\partial\Omega_F$, $\partial\Omega_v$, $\partial\Omega_c$.

Here $\partial\Omega_F$ is the part of the surface where the forces $\mathbf{F} = \mathbf{F}^*$ are given; $\partial\Omega_v$ is the part of the surface where the velocities $\mathbf{v} = \mathbf{v}^*$ are given; $\partial\Omega_c$ is the area of contact with the instrument, on which the kinetic constraint and condition of friction described below apply: (a) for the field $\mathbf{v}$ and

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velocity of the instrument **w**, the normal components are equal; (b) the tangential components $\mathbf{f}_t$ of the force **F** satisfies the condition $|\mathbf{F}_t| \leqslant k = \text{const}$, where if $|\mathbf{F}_t| < k$ at a given point, then $\mathbf{v}_t = \mathbf{w}_t$ (sticking); but if $|\mathbf{F}_t| = k$, then the vector $(\mathbf{v}_t - \mathbf{w}_t)$ is in the opposite direction to $\mathbf{F}_t$ (sliding).

2. VARIATIONAL FORMULATION

Let

$$A = \{ \mathbf{v} \in H^1(\Omega): \ v = v_* \ \text{на} \ \partial\Omega_v, \ v_n = w_n \ \text{on} \ \partial\Omega_c \}$$
$$M = \{ \mathbf{v} \in H^1(\Omega): \ \operatorname{div}(v) = 0 \}$$

Let (v^0, σ^0) be a solution of boundary-value problem 1. Then [1] v^0 is a solution of the following variational problem.

Problem 2. It is required to find the field v^0 which, on the set $A \cap M$, gives a minimum of the functional

$$J(v) = \tfrac{1}{2}\mu \int_\Omega |e(v)|^2 dx + \tau_s \int_\Omega |e(v)|\, dx - \int_{\partial\Omega_F} (F^*, v)\, dS + k \int_{\partial\Omega_c} |v_t - w_t|\, dS$$

We shall assume that the kinematic conditions do not allow Ω to move as an absolutely solid body, that is, if the field v is the difference of the fields from A and $e(v) \equiv 0$, then $v \equiv 0$. On this assumption [1], it can be stated that the solution of Problem 2 exists and is unique.

3. SADDLE POINT

The difficulty that arises in solving Problem 2 is that a minimum of the functional J must be sought for fields v satisfying the incompressibility condition $v \in M$, and not over the whole set A. This difficulty is removed [2] by introducing Lagrange multipliers. Let $p \in L^2 = L^2(\Omega)$. We put

$$G(v, p) = J(v) + \int_\Omega p \operatorname{div}(v)\, dx$$

Problem 3. It is required to find a saddle point (v^0, p^0) of the function G on the set $A \times L^2$, that is

$$G(v^0, p^0) = \min_{v \in A} \sup_{p \in L^2} G(v, p) = \max_{p \in L^2} \inf_{v \in A} G(v, p)$$

It can be verified that Problem 3 has a unique solution (v^0, p^0). The field v^0 is the solution of Problem 2. In fact, since

$$G(v^0, p^0) = J(v^0) + \sup_{p \in L^2} \int_\Omega p \operatorname{div}(v^0)\, dx < +\infty$$

we have $\operatorname{div}(v^0) = 0$, and $v^0 \in M$. Thus

$$J(v^0) = G(v^0, p^0) = \min_{v \in A} G(v, p^0) \leqslant \min_{v \in A \cap M} G(v, p^0) = \min_{v \in A \cap M} J(v)$$

4. STRESS TENSOR

Since $v^0 \in H^1(\Omega)$, the stress tensor $\sigma^0 \in T(\Omega)$ and it is therefore impossible to introduce the density of surface forces F on $\partial\Omega$ with the formula $F_i = \sigma_{ij}n_j$. A weak formulation of the boundary condition for forces is therefore needed. We put

$$R(\Omega) = \{\sigma = \{\sigma_{ij}\} \in T(\Omega)\colon\ \partial\sigma_{ij}/\partial x_j = 0 \qquad i = 1, 2, 3\}$$

If σ has continously differentiable components and satisfies the equilibrium equations, then

$$\int_\Omega \langle \sigma, e(v)\rangle\, dx = \int_{\partial\Omega} \sigma_{ij}n_jv_i dS, \quad \forall v \in H^1(\Omega)$$

Using this equation, it can be shown [3] that there exists a unique continuous linear operator

$$\nu\colon\ R(\Omega) \to H^{-1/2}(\partial\Omega)$$

such that if the tensor σ has continuous components, the function $\nu(\sigma)$ operates subject to the formula

$$\nu(\sigma)\,(u) = \int_{\partial\Omega} (F, u)\, dS, \quad \forall u \in H^{1/2}(\partial\Omega); \quad F = \{F_i\}, \quad F_i = \sigma_{ij}n_j$$

We shall call $\nu(\sigma)$ the force density on $\partial\Omega$ corresponding to the tensor σ. For $\nu(\sigma)$ it is reasonable to introduce $\nu_i(\sigma) \in H^{-1/2}(\partial\Omega)$ such that

$$\nu(\sigma)\,(u) = \sum_{i=1}^{3} \nu_i(\sigma)\,(u_i), \quad \forall u = \{u_i\} \in H^{1/2}(\partial\Omega)$$

(We write $\nu(\sigma) = \{\nu_i(\sigma)\}$, and call $\nu_i(\sigma)$ a component of $\nu(\sigma)$.) We also introduce functionals $\nu_n(\sigma)$ and $\nu_t(\sigma)$, which we call the normal and tangential components.

The boundary conditions on the forces in Problem 1 will be understood in a general sense, that is, the functional $\nu(\sigma)$ is used instead of density functions. The following assertion is proved in the usual way.

Assertion. Let (v^0, p^0) be a saddle point of Problem 3. Then a deviator $s^0 \in \partial\varphi(e(v^0))$ exists such that the pair (v^0, σ^0), where $\sigma^0 = \{\sigma^0_{ij}\}$, $\sigma^0_{ij} = s^0_{ij} + p^0\delta_{ij}$ (δ_{ij} is the Kronecker delta) is a (generalized) solution of boundary-value problem 1.

5. THE UZAWA ALGORITHM

The saddle point of Problem 3 can be found using the Uzawa algorithm [4]. For fixed p we find the minimum with respect to v of the functional

$$G(v, p) = J(v) + \int_\Omega p \operatorname{div}(v)\, dx$$

The functional J is non-differentiable, making minimization difficult. We therefore make a slight modification to Problem 3. Let

$$Q = \{q = \{q_{ij}\} \in T(\Omega)\colon |q(x)| \leq \tau_s \text{ almost everywhere in } \Omega\}$$
$$R = \{r = \{r_i\} \in L^2(\partial\Omega_c)\colon\ |r(x)| \leq k \text{ almost everywhere in } \partial\Omega_c\}$$

We put $Z = L^2(\Omega) \times T(\Omega) \times L^2(\Omega_c)$, $B = L^2(\Omega) \times Q \times R \subset Z$. Let

$$L(v, z) = \tfrac{1}{2}\mu \int_\Omega |e(v)|^2 dx + \int_\Omega \langle q, e(v)\rangle\, dx + \int_\Omega p \operatorname{div}(v)\, dx - \int_{\partial\Omega_F} (F^*;\ v)\, dS +$$
$$+ \int_{\partial\Omega_c} (r, v_t - w_t) dS, \qquad v \in A, \quad z = (p, q, r) \in Z$$

Instead of Problem 3, consider the following.

Problem 4. It is required to find a saddle point (v^0, z^0) of the functional L on the set $A \times B$, that is

$$L(v^0, z^0) = \min_{v \in A} \sup_{z \in B} L(v, z) = \max_{z \in B} \inf_{v \in A} L(v, z)$$

If (v^0, z^0), $z^0 = (p^0, q^0, r^0)$ is a solution of Problem 4, from the equations

$$\tau_s \int_\Omega |e(v)|\, dx = \max_{q \in Q} \int_\Omega \langle q, e(v)\rangle\, dx, \quad k \int_{\partial\Omega_c} |v_t - w_t|\, dS = \max_{r \in R} \int_{\partial\Omega_c} (r, v_t - w_t)\, dS$$

it follows that (v^0, p^0) is a solution of Problem 3. It can also be seen that q^0 is the deviator and $\{p^0 \delta_{ij}\}$ is the spherical part of the real stress tensor, that is, the solution of Problem 1 can be found by solving Problem 4.

We will describe the Uzawa algorithm for Problem 4. Let Φ: $Z \to A$ be an operator such that $v = \Phi(z)$ is a minimum point on the set A of the functional $L(v, z)$ with respect to v for given z. The algorithm is as follows. We choose an arbitrary initial value $z^{(1)} \in B$. Each step of the process can be described as follows:

1. for fixed $z^{(n)} \in B$, find $v^{(n+1)} = \Phi(z^{(n)})$,
2. the next value $z^{(n+1)} = (p^{(n+1)}, q^{(n+1)}, r^{(n+1)})$ is computed using the formulae: $p^{(n+1)} = p^{(n)} + \rho \operatorname{div} v^{(n)}$; $q^{(n+1)}$ is the projection on Q of the element $q^{(n)} + \rho e(v^{(n)})$; $r^{(n+1)}$ is the projection on R of the element $r^{(n)} + \rho(v_t^{(n)} - w_t)$. The number $\rho \in (0, \rho_{\max})$ and there is a limit for $\rho_{\max}$.

Thus, the algorithm essentially consists of computing values of the operator Φ. $v = \Phi(p, q, r)$ is calculated in two stages: v on $\partial\Omega$ is first found using an integral equation, and then v inside Ω is calculated from an explicit equation. Suppose first that p, q and r are continuously differentiable. Since $v = \{v_i\}$ is the point of the minimum of functional L on A for fixed p, q and r, for any permissible variation of the field $\zeta = \{\zeta_i\}$, that is $\zeta \in \mathbf{H}^1(\Omega)$, $\zeta = 0$ on $\partial\Omega_v$, ζ_n on $\partial\Omega_c$, we have

$$I_1 + I_2 + I_3 - \int_{\partial\Omega_F} (F^*, \zeta)\, dS + \int_{\partial\Omega_c} (r, \zeta_t)\, dS = 0 \tag{1}$$

$$I_1 = \int_\Omega \mu \langle e(v), e(\zeta)\rangle\, dx, \quad I_2 = \int_\Omega \langle q, e(\zeta)\rangle\, dx, \quad I_3 = \int_\Omega p \operatorname{div}(\zeta)\, dx$$

Integrating by parts, we obtain

$$I_1 = \int_{\partial\Omega} \mu e_{ij}(v) n_j \zeta_i dS - \int_\Omega \mu \frac{\partial e_{ij}(v)}{\partial x_j} \zeta_i dx$$

$$I_2 = \int_{\partial\Omega} q_{ij} n_j \zeta_i dS - \int_\Omega \frac{\partial q_{ij}}{\partial x_j} \zeta_i dx, \quad I_3 = \int_{\partial\Omega} p \zeta_i n_i dS - \int_\Omega \frac{\partial p}{\partial x_i} \zeta_i dx$$

From (1) it follows that

$$\mu \frac{\partial e_{ij}(v)}{\partial x_j} + b_i = 0, \quad \left(b_i = \frac{\partial q_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_i} \right) \tag{2}$$

$$F = F^* \text{ on } \partial\Omega_F, \quad F_t = -r \text{ on } \partial\Omega_c \, (F = F_i\ ,\ F_i = \mu e_{ij}(v) n_j + q_{ij} n_j + p n_i) \tag{3}$$

Equations (2) are of the form of equilibrium equations for an elastic medium with modulus of elasticity for shear $G = \mu/2$ and Poisson's ratio $\nu = 0$ subjected to a force with volume density b_i.

Let

$$u^{(k)}(\xi, x) = \{u_i^{(k)}(\xi, x)\}, \quad F^{(k)}(\xi, x) = \{F_i^{(k)}(\xi, x)\}, \quad k = 1, 2, 3$$

$$u_i^{(k)}(\xi, x) = \frac{1}{8\pi\mu r} \left(3\delta_{ik} + \frac{r_i r_k}{r^2}\right)$$

$$F_i^{(k)}(\xi, x) = -\frac{1}{8\pi\mu r^2} \left\{ \left(\delta_{ik} + \frac{r_i r_k}{r^2}\right) \frac{\partial r}{\partial n} - \frac{r_i n_k - r_k n_i}{r} \right\}$$

$$r = (r_i r_i)^{1/2}, \quad r_i = x_i - \xi_i$$

Then [5] (putting $G=\mu/2$ and $\nu=0$) we find that $u^{(k)}(\xi, x)$ is a fundamental solution of Eq. (3) and $F_i^{(k)}=\sigma_{ij}^{(k)}n_j$ on the set $\partial\Omega$, where $\sigma_{ij}^{(k)}=\mu e_{ij}(u^{(k)})$. From Eq. (2) it follows that

$$I_5+I_6=0 \tag{4}$$

$$I_5=\int_\Omega \mu \frac{\partial e_{ij}(v)}{\partial x_j} u_i^{(k)}(\xi, x)\, dx, \qquad I_6=\int_\Omega b_i(x) u_i^{(k)}(\xi, x)\, dx$$

We have

$$I_5=\int_{\partial\Omega} \mu e_{ij}(v) n_j u_i^{(k)} dS(x)-\int_\Omega \mu e_{ij}(v) e_{ij}(u^{(k)})\, dx=$$

$$=\int_{\partial\Omega} \mu e_{ij}(v) n_j u_i^{(k)} dS(x)-\int_{\partial\Omega} F_i^{(k)} v_i dS(x)+\int_\Omega \mu \frac{\partial e_{ij}(u^{(k)})}{\partial x_j} v_i\, dx$$

$$I_6=\int_\Omega \left[\frac{\partial q_{ij}}{\partial x_j}+\frac{\partial p}{\partial x_i}\right] u_i^{(k)} dx=\int_{\partial\Omega}(q_{ij}n_j+pn_i)u_i^{(k)} dS(x)-$$

$$-\int_\Omega q_{ij} \frac{\partial u_i^{(k)}}{\partial x_j}\, dx-\int_\Omega p \frac{\partial u_i^{(k)}}{\partial x_i}\, dx$$

Since $u^{(k)}$ is a fundamental solution, for $\forall\xi\in \mathrm{int}\,\Omega$ the last integral in the expression for I_5 is equal to $-v_k(\xi)$. Thus, from (4) and the definition of F in (3) for $\xi\in\mathrm{int}\,\Omega$ we have

$$v_k(\xi)=\int_{\partial\Omega} F_i(x) u_i^{(k)}(\xi, x) dS(x)-\int_{\partial\Omega} F_i^{(k)}(\xi, x) v_i(x)\, dS(x)-$$

$$-\int_\Omega q_{ij}(x) \frac{\partial u_i^{(k)}(\xi, x)}{\partial x_j}\, dx-\int_\Omega p(x) \frac{\partial u_i^{(k)}(\xi, x)}{\partial x_i}\, dx \tag{5}$$

Integral equations for $v(\xi)$ on $\partial\Omega$ are obtained similarly

$$c_{ki}(\xi) v_i(\xi)=\int_{\partial\Omega} F_i u_i^{(k)} dS(x)-\int_{\partial\Omega} F_i^{(k)} v_i dS(x)-\int_\Omega q_{ij} \frac{\partial u_i^{(k)}}{\partial x_j}\, dx-$$

$$-\int_\Omega p \frac{\partial u_i^{(k)}}{\partial x_i}\, dx, \quad k=1,2,3 \tag{6}$$

Formulae for $c_{ki}(\xi)$ exist [5]. In particular, at smooth points $c_{ki}(\xi)=\delta_{ki}/2$.

Thus, the first stage in the calculation of $v=\Phi(p, q, r)$ is to find fields $v=\{v_i\}$, $F=\{F_i\}$ on $\partial\Omega$ such that the integral equations (6) apply and $v=v^*$ on $\partial\Omega_v$, $F=F^*$ on $\partial\Omega_F$, $F_t=-r$ on $\partial\Omega_c$. We then find v in $\mathrm{int}\,\Omega$ using formula (5).

The calculation of $v=\Phi(p, q, r)$ has been examined for the case where p, q and r are continuously differentiable. Since the operator Φ is continuous, the calculation is also valid in the general case.

Note. If the values of the operator Φ are calculated in the usual way, discrete approximation of the problem leads to the finite element method. The method described above leads to the boundary element method.

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